MATHEMATICAL SCIENCES

INTERVAL EDGE-COLORING OF EVEN BLOCK GRAPHS

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Abstract

An edge-coloring of a graph G with consecutive integers $c_1, \dots, c_t$ is called an interval t-coloring, if all colors are used, and the colors of edges incident to any vertex of G are distinct and form an interval of integers. A graph G is interval colorable, if it has an interval t-coloring for some positive integer t. For an interval colorable graph G, $w(G)$ denotes the smallest value of t for which G has an interval t-coloring. A block graph is a type of graph in which every biconnected component (block) is a clique. Even block graphs are a subclass of block graphs where each block has an even number of vertices. In this paper we show that any even block graph has an interval edge-coloring with $w(G) \leq 2 \cdot (\Delta(G) - 1)$. Furthermore, we show that this upper bound can not be improved for even block graphs.

Keywords: block graph, interval t-coloring, interval edge-coloring, even block graph

Introduction

All graphs considered in this paper are undirected (unless explicitly said), finite, and have no loops or multiple edges. For an undirected graph G, let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$. Let T be a tree (a connected undirected acyclic graph). Let $d_T(u, v)$ be the length of the path from the vertex u to the vertex v. Since T is a tree, there is exactly one path connecting two vertices.

For a directed graph $\vec{G}$ if there is an edge from a vertex u to a vertex v we will denote it as $u \to v$. The graph G is called the underlying graph of a directed graph $\vec{G}$ if $V(G) = V(\vec{G})$ and $E(G) = \{\vec{G} | iff\ u \to v\ or\ v \to u\}$ (between any pair of vertices u and v, if the directed graph has an edge $u \to v$ or an edge $v \to u$, the underlying graph includes the edge (u, v)).

For a tree T and a vertex r, let T_r be the directed graph whose underlying graph is T and in T_r each edge is directed in such a way that for each vertex $v \in V(T_r)$ there is a path in T_r from r to v. We will say that T_r is a rooted tree with a root r. Fig. 1 illustrates the rooted tree T_{v_1} with the root v_1.

Let T_r be a rooted tree, the depth of a vertex v, denoted by $h(v)$, is the length of the unique path from the root r to the vertex v. A vertex u is said to be the parent of the vertex v, denoted by $p(v)$, if $u \to v$. In that case the vertex v is said to be a child of the vertex u. The children of a vertex $v \in V(T_r)$ are the set $W \subseteq V(T_r)$ of all vertices w of the tree T_r satisfying the condition $v \to w$. A vertex having no children is said to be a leaf vertex. Non-root two vertices $a, b \in V(T_r)$ are said to be sibling vertices if $p(a) = p(b)$. For a vertex v let $S(v)$ be the subtree induced by all the vertices w such that there is a path from v to w in T_r [1].

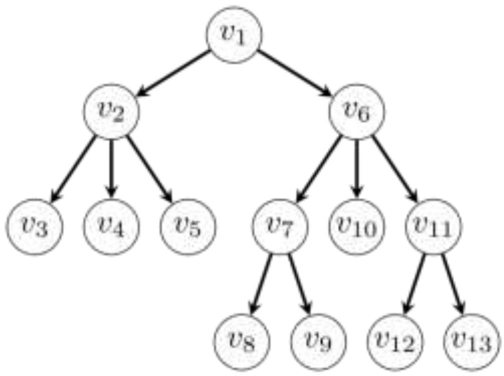

Fig. 1. *A rooted tree T_{v_1} with the root v_1.*

A cut vertex is any vertex whose removal increases the number of connected components. Any connected graph decomposes into a tree of biconnected components called the block-cut tree of the graph. A block graph is a type of graph in which every biconnected component (block) is a clique (every two distinct vertices in the clique are adjacent). Fig. 2 illustrates an example of a block graph and its respective block-cut tree.

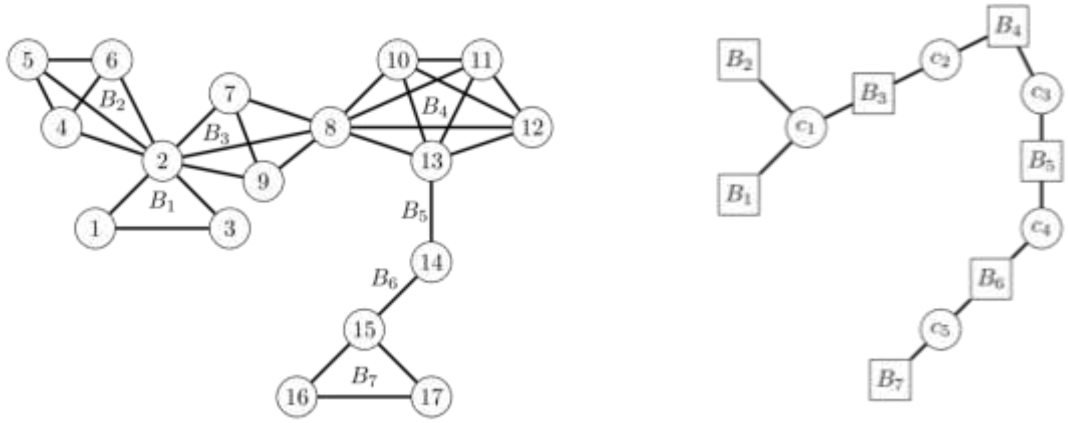

Fig. 2. Block graph on the left and its respective block-cut tree on the right.
Even block graphs are a subclass of block graphs where each block has an even number of vertices.

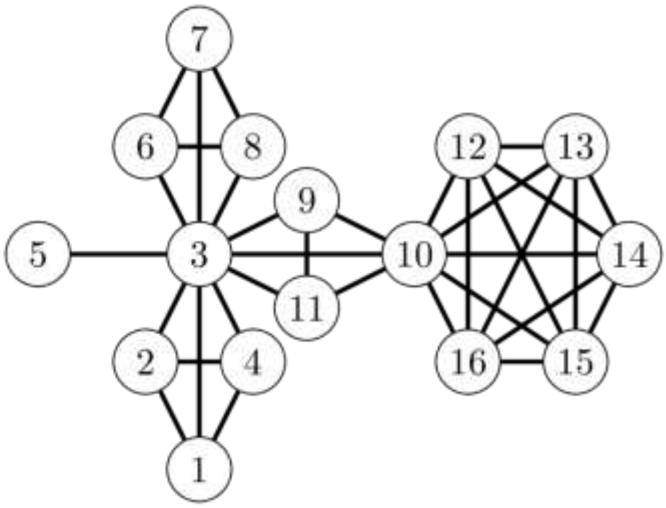

Fig. 3. An example of an even block graph. Each block is a clique with an even number of vertices.

An edge-coloring of a graph G is an assignment of colors to the edges of the graph so that no two adjacent edges have the same color. An edge-coloring of a graph G with colors $1, \dots, t$ is an interval t-coloring if all colors are used, and the colors of edges incident to each vertex of G form an interval of integers. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. The set of all interval colorable graphs is denoted by $\mathfrak{N}$. The concept of an interval edge-coloring of a graph was introduced by Asratian and Kamalian [2]. This means that an interval t-coloring is a function $\alpha: E(G) \to \{1, \dots, t\}$ such that for each edge e the color $\alpha(e)$ of that edge is an integer from 1 to t, for each color from 1 to t there is an edge with that color and for each vertex v all the edges incident to v have different colors forming an interval of integers. The set of integers $\{a, a+1, \dots, b\}$, $a \le b$, is denoted by $[a, b]$. Let I_k be the set $[1, k]$ of integers, then 2^{I_k} is the set of all the subsets of I_k. We will denote by $\tau(I_k)$ the set of all the elements from 2^{I_k} that are an interval of integers. More formally $\tau(I_k) = \{s: s \in 2^{I_k}, s\ is\ a\ non\ empty\ interval\ of\ integers\}$.

For an interval coloring, α and a vertex v, the set of all the colors of the incident edges of v is called the spectrum of that vertex in α and is denoted by $S_\alpha(v)$. The smallest and the largest numbers in $S_\alpha(v)$ are denoted by $\underline{S_\alpha}(v)$ and $\overline{S_\alpha}(v)$, respectively.

Interval edge-colorings have been intensively studied in different papers. In [3] it was shown that every tree is from $\mathfrak{N}$. Lower and upper bounds on the number of colors in interval edge-colorings were provided in [4] and the bounds were improved for different graphs: planar graphs [5], r-regular graphs with at least $2 \cdot r + 2$ vertices [6], cycles, trees, complete bipartite graphs [3], n-dimensional cubes [7,8], complete graphs [9, 10], Harary graphs [11], complete k-partite graphs [12]. In [13], interval edge-colorings with restrictions were considered, in which case there can be restrictions on the edges for the allowed colors.

Interval edge-coloring of even block graphs

Here we consider the following two problems for even block graphs.

Problem 1: Interval edge-coloring

Given a graph G. Find any interval edge-coloring. More formally, find an edge-coloring $\alpha: E(G) \to \mathbb{N}$ such that for each vertex v all the edges incident to v have different colors forming an interval of integers.

Problem 2: Interval edge-coloring with minimal colors

Given a graph G. Find the minimal t for which there is an interval t-coloring. More formally, find an edge-coloring $\alpha: E(G) \to [1, t]$ (with minimal t)such that for each color from 1 to t, there is an edge with that color, and for each vertex v all the edges incident to v have different colors forming an interval of integers.

First, we will solve the Problem 1 for even block graphs by providing an upper bound for $w(G)$. Then we will show that the upper bound can not be improved. This way, we will get close to the solution of the Problem 2 for even block graphs.

It is known that all trees have interval edge-coloring. A complete graph K_n has an interval edge-coloring if and only if n is even. The class of even block graphs contains the class of even complete graphs and the class of trees (since every block in a tree is an edge and contains two vertices). Let us first show how to color the edges of a complete graph to have an interval edge coloring.

Lemma 1: For a complete graph $K_{2\cdot m}$, and for any positive integers l, r with $l \le r$ and $r - l + 1 = 2 \cdot m - 1$, there is an interval edge-coloring α such that $S_\alpha(v) = [l, r]$ for all $v \in V(K_{2\cdot m})$.

Let $N = 2 \cdot m$. From [4] it is known that $w(K_{2\cdot m}) = 2 \cdot m - 1 = \Delta(K_{2\cdot m})$. It means there is an interval t-coloring with $t = 2 \cdot m - 1$. Adding $l - 1$ to all the edges of an interval t-coloring will produce an interval edge-coloring with colors from l to $l + (2 \cdot m - 1) - 1 = r$. When it comes to the coloring, assume the vertices are labeled from 1 to N. Coloring the edge $(i, j), 1 \le i, j \le N$ with the color $l + ((i + j)\%(N - 1)$

will ensure that all the spectrums are $[l, r]$.

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Unlike complete graphs, it is not true that if a block graph has a block with an odd number of vertices, then it does not have interval edge-coloring. Fig. 4 illustrates such a counterexample.

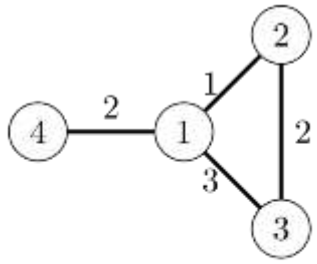

Fig. 4. An example of a block graph that has interval edge-coloring.

We will show that every even block graph has an interval edge-coloring.

Theorem 1: Every even block graph G has an interval edge-coloring. Furthermore $w(G) \le 2 \cdot (\Delta(G) - 1)$ when $\Delta(G) \ge 2$.

If the graph G is a complete graph with an even number of vertices, then from Lemma 1 $w(G) = \Delta(G)$. When $\Delta(G) \ge 2$, $2 \cdot (\Delta(G) - 1) \ge \Delta(G)$ hence the statement is valid.

We can assume that G is not a complete graph. It means the graph G has at least one cut vertex. Let $t = 2 \cdot (\Delta(G) - 1)$. Let r be an arbitrary cut vertex and consider the rooted block-cut tree T_r of the graph G. We will provide a top-down approach for coloring the graph edges. For the cut vertices of the block-cut tree, we will color the edges of that vertex and all the blocks to which it belongs. Every block B of the graph G will be colored with some $[l, r]$ interval $r - l + 1 = |V(B)| - 1$. Since all the blocks have even number of vertices, from Lemma 1 it is possible to have such interval edge-coloring. At every point, the colored edges will form an interval of integers for every vertex (even if some of the edges for that vertex are not colored yet). For a block vertex $v \in V(T_r)$, let $B(v)$ be the block of the graph G.

For the vertex r we can color its incident edges with the colors from 1 to $d_G(r)$. Let $u_1, \dots, u_k$ ($k = d_{T_r}(r)$) be the child vertices of r in T_r. Since r is a cut vertex, $B(u_1), \dots, B(u_k)$ are the blocks containing the vertex r. Let $s_i = \left|V\left(B(u_i)\right)\right| - 1$ for $1 \le i \le k$. If we color the block $B(u_1)$ with colors $[1, s_1]$, the block $B(u_2)$ with colors $[s_1 + 1, s_1 + s_2]$, ... , the block $B(u_k)$ with colors $[s_1 + s_2 + \dots + s_{k-1} + 1, s_1 + s_2 + \dots + s_k]$ using the Lemma 1 then the spectrum of the vertex r will be the interval $[1, d_G(r)]$ and all the block that contain the vertex r will have an interval edge-coloring.

Now consider any non-root vertex $v \in V(T_r)$. The vertex v is either a cut vertex or a block vertex in T_r.

If the vertex v is a cut vertex, then its parent vertex $p(v)$ is a block vertex. Since we already colored the edges of the block $B(p(v))$, the colors of the edges incident to v that belong to that block form an interval of integers. The length of the interval is equal to $a = \left|V\left(B\big(p(v)\big)\right)\right| - 1$ and assume it is the interval $[l, r], (r - l + 1 = a, [l, r] \subseteq [1, t]$. Since v is a cut vertex, it belongs to more than one block, which means it has edges that do not belong to the block $B(p(v))$, hence $1 \le a \le \Delta(G) - 1$.

Now we need to color the rest of the edges incident to v so that each of the blocks that contain the vertex v are colored with intervals. Let $u_1, \dots, u_k$ be the child vertices of v and let $B(u_1), \dots, B(u_k)$ be their respective blocks. Let $s_i = \left|V\left(B(u_i)\right)\right| - 1$ which is the number of edges from the block $B(u_i)$ incident to the vertex v. Let $b = \sum_{i=1}^{k} s_i$ then

$$a + b = \left|V\left(B\big(p(v)\big)\right)\right| - 1 + \sum_{i=1}^{k}(\left|V\left(B(u_i)\right)\right| - 1) = d_G(v) \le \Delta(G).$$

Assume that we can find an interval $[c, d] \subseteq [1, t]$ of length b such $[c, d] \cap [l, r] = \emptyset$ and $[c, d] \cup [l, r] \in \tau(I_t)$. Then the vertex v would be colored with interval of integers and we could take the interval $[c, d]$ and split it into smaller intervals of lengths $s_1, \dots, s_k$ to color the blocks corresponding to the children of v. Using the Lemma 1, the block $B(u_1)$ could be colored with colors $[c, c+s_1-1]$, the block $B(u_2)$ with colors $[c+s_1, c+s_1+s_2-1]$, ..., the block $B(u_k)$ with colors $[c+s_1+\cdots+s_{k-1}, c+s_1\cdots+s_k-1] = [d-s_k+1, d]$ since $b=\sum_{i=1}^{k} s_i$.

Now let us show that we can always find such $[c,d]$ interval. If $l-b \geq 1$ then $[c,d]=[l-b, l-1]$. If $l \leq b$ we need to show that $r+b \leq t$ in which case we could take $[c,d]=[r+1, r+b]$. Since $1 \leq a \leq \Delta(G)-1$ and $a+b \leq \Delta(G)$ we have $1 \leq b \leq \Delta(G)-1$.

$r+b = l+a-1+b \leq b+a-1+b$ and we know that $a+b \leq \Delta(G)$, $b \leq \Delta(G)-1$ we can get $b+a-1+b \leq \Delta(G)-1+\Delta(G)-1 = t$.

The intuition is that as long as t is greater or equal than $2\cdot(b-1)+a+1$ we can always find an interval of length b to the left or the right of any given interval that has length a. The maximal value will be achieved if $a=1, b=\Delta(G)-1$ in which case we get $2\cdot(\Delta(G)-2)+2$.

If the vertex v is a block vertex, then we already know how the edges of the block $B(v)$ are colored since the parent vertex $p(v)$ is a cut vertex. Using the fact that $B(v)$ has even number of vertices and is a clique we colored with some colors $[l,r]$ such that $r-l+1=|V(B(v))|-1$ using the Lemma 1. We can continue coloring the children cut vertices of the vertex v.

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Theorem 2. For every even integer m there exist an even block graph G with $\Delta(G)=m$ for which $(G) \geq 2\cdot(m-1)$.

Let $m=2\cdot k$ be an even integer. When $m=2$ we have $2\cdot(m-1)=2=m$ and since we should have $\Delta(G)=m$ and it is known that $w(G) \geq \Delta(G)$ the statement is obvious for this case. We can assume $m \geq 4$ in which case $2\cdot(m-2)+1 \geq m+1$.

We need to construct an even block graph G such that $\Delta(G)=m$ and there is no interval edge-color with $2\cdot(m-2)+1$ colors. Consider the graph illustrated in Fig. 5.

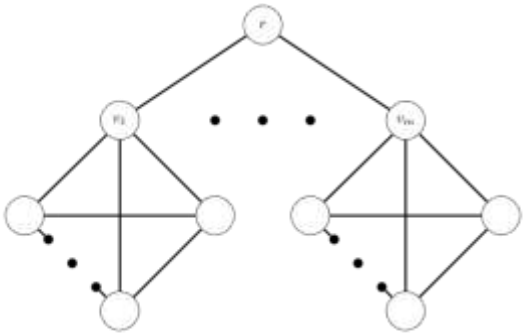

Fig. 5. An even block graph where the vertex r is connected to the vertices $v_1, \ldots, v_m$. Each of the vertices $v_1 \ldots, v_m$ are inside blocks that are cliques having m vertices.

In that graph, there is a root vertex r that is connected to m vertices $v_1, \ldots, v_m$ and each of these m vertices belong to a clique C_i each of which has m vertices. The constructed graph has m^2+1 vertices. Let $t=2\cdot(m-2)+1$ and suppose there is an interval edge-coloring α such that all the colors are from 1 to t. The colors of the edges $(r,v_1), \ldots, (r,v_m)$ form an interval of integers $[a,b]$ with length m $(b-a+1=m)$.

Consider the color $m-1$ which is the middle color of the interval $[1,t]$. Since $[a,b] \subseteq [1,t]$ $(t \geq m+1)$ it means $b \geq m$ and $a \leq t-m+1 = m-2$ which means the color $m-1 \in [a,b]$. Let v_i be the vertex for which $\alpha(r,v_i)=m-1$ and consider the clique C_i that contains the vertex v_i. If we take the edges of the clique that have color $m-1$, they will form a matching and will not contain the vertex v_i, since it is already connected to the vertex r with the color $m-1$. Since $|V(C_i)|=m=2\cdot k$ there is a vertex $u \in V(C_i)$ which does not have an incident edge with color $m-1$. Fig. 6 illustrates that case.

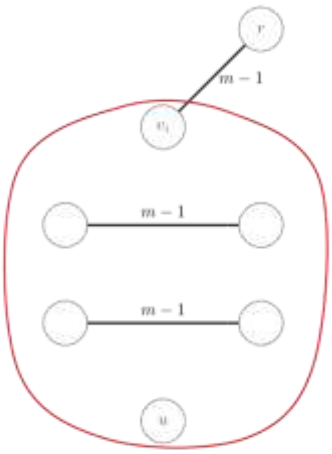

Fig. 6. The edges of a clique containing the vertex v_i having color $m-1$. There is a vertex u that does not have an incident edge with color $m-1$.

For the vertex u, $d_G(u)=m-1$ and the spectrum $S_\alpha(u)=[c,d]$ has length $m-1$ ($c-d+1=m-1$). This means that either $c\geq m$ or $d\leq m-2$. If $c\geq m$ then $d\geq m+(m-1)-1=2\cdot(m-1)\geq t+1$ which is a contradiction. If $d\leq m-2$ then $c\leq d-(m-1)+1=0$ which is a contradiction. This means there is no interval edge-coloring with $t=2\cdot(m-2)+1$ colors hence $w(G)\geq 2\cdot(m-1)$.

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Corollary: If G is an even block graph then $w(G)\leq 2\cdot(\Delta(G)-1)$ and the inequality can not be improved for the class of even block graphs.

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